A Generalized Kac Model as a Dynamical System *

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Ising chains characterized by smoothly varying two-body potentials, including a generalized version of the Kac model, are studied. As a novel aspect we show that one-dimensional maps associated with such chains in the spirit of the thermodynamical formalism possess fractal derivatives. We discuss how long-range potentials lead to intermittent-like behavior in the associated one-dimensional map.

The Kac model is a one-dimensional Ising type chain with exponentially decreasing long-range interaction [1]. As an illustrative example of Ising chains with smooth two-body interactions we study a generalized version of the Kac model defined by the real valued Hamiltonian

$$\mathcal{H} = -\tfrac{1}{2}\sum_{i\neq j}\sigma_i\sigma_j J(|i-j|) \tag{1}$$

with coupling

$$J(x) = \tfrac{1}{2}(\lambda^x + \lambda^{*x}) = |\lambda|^x \cos\varphi\, x, \quad 0<|\lambda|<1. \tag{2}$$

Here φ denotes the phase of the complex interaction parameter λ, and $\sigma_i = \pm 1$ stands for the spin variable at site i. The interaction energy between two sites at distance x decays exponentially with x but now with an oscillating amplitude of wavelength $2\pi/\varphi$. This oscillation mimics the effect of the indirect exchange interaction between spins mediated by electrons that leads in realistic models to RKKY type interaction with an algebraic decay [3]. The generalized Kac model, (1, 2), has been introduced in [2]. Here we present some results concerning the dependence on the phase φ not published before and discuss them in a broader context.

The basic observation of our approach is that thermal properties of the Kac chains (such as free energy, magnetization, heat capacity etc.) can be obtained from a simple functional representation of the transfer matrix [4]. For the generalized model the transfer matrix $\mathcal{L}$ can be represented in the space of holomorfic functions of the complex variable z and its conjugate z^* by the operator [2]

$$\mathcal{L}\,g(z,z^*) = \sum_{\sigma=\pm 1} \cdot \exp\left[\frac{\beta\sigma}{2}(z+z^*)\right] g(\lambda\sigma+\lambda z, \lambda^*\sigma+\lambda^* z^*), \tag{3}$$

where β is the inverse temperature.

The largest eigenvalue of the transfer operator $\mathcal{L}$ can be obtained by letting it act on a smooth function and extracting the growth rate. Then the free energy $\beta F(\beta)$ reads as the negative logarithm of the largest eigenvalue. In the generalized case the free energy depends on the complex interaction parameter λ. In Fig. 1 we have plotted the free energy contours at a given temperature on the complex $|\lambda|<1$ disk. As starting function we choose $g(z,z^*)\equiv 1$. Due to the exponentially fast convergence an error of less than 10^{-2} can be reached at a number $n=14$ of iterations for $|\lambda|<0.9$. The complicated structure of the contours can be understood by recalling that the interaction energy between two spins σ_i and σ_{i+x} is proportional to $J(x)$ defined by (2). When $|\lambda|\ll 1$, the nearest-neighbor interactions dominate and the free energy is close to that of an Ising chain with coupling constant $|\lambda|\cos(\varphi)$. For larger values of λ longer and longer interactions play a dominant role and the total interaction energy appears as a kind of Fourier series leading to interference-like patterns in the free energy. We have also found that the relatively large values of the free energy are accompanied by frustration, a typical effect in disordered systems.

The transfer operator in representation (3) has a surprisingly similar structure to the so-called general-

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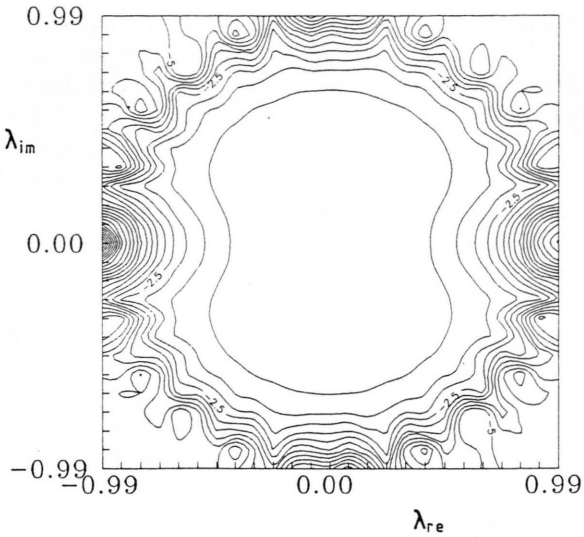

Fig. 1. Equipotential lines on the free energy surface $\beta F(\beta)$ at $\beta = 1.2$ on the $|\lambda| < 1$ disk in the range $-5 < \beta F(\beta) < -1$. The free energy was obtained from the growth rate of $\mathscr{L}^n 1$ after $n = 14$ iterations, where operator $\mathscr{L}$ is defined by (3). Level lines $\beta F(\beta) = -5, -4.5, \ldots, -1$ are plotted.

ized Frobenius-Perron (FP) operator [5], which has been used to characterize scaling properties of one-dimensional maps. For a single-humped map $f(x)$ with inverse branches $F_\varepsilon(y)$, $\varepsilon = 0, 1$, the generalized Frobenius-Perron operator H is given by

$$H\psi(x) = \sum_{\varepsilon=0,1} |F'_\varepsilon(x)|^\beta \psi(F_\varepsilon(x)), \tag{4}$$

where β is a weighting parameter and prime denotes differentiation. The free energy [5] $\beta \mathscr{F}(\beta)$ of the map follows from the negative logarithm of the largest eigenvalue of H. Despite the striking similarity, the two operators (3) and (4) cannot be made equivalent by smooth transformations. This naturally arises the question of how to find a one-dimensional map with the same free energy as a given spin chain with two-body interaction.

To construct such an associated map we have to find a unique correspondence between the microstates of the spin chain of length n and the symbolic codes of length n in the dynamics. Since the Kac chain contains two-state spins, the associated map should have a binary encoding. Among several possibilities we restrict ourselves to maps topologically similar to the Bernoulli shift. Identifying the cylinder lengths in the generating partition of the map with the Boltzmann factors of the corresponding microstates, we have found [2] that the derivatives of the map are given as

$$F'_\varepsilon(y) = \exp[-E_0 + u_\varepsilon(y)]. \tag{5}$$

Here E_0 is an energy constant, and instead of the binary symbolic codes of the cylinders their real number representation $y = \sum \varepsilon_i 2^{-i}$ has been introduced. The quantities $u_\varepsilon(y) \sim \varepsilon \sum_{i=1}^n \varepsilon_i J(i)$, called *symbolic interaction functions*, represent the interaction energy of the last spin ε added at site $n+1$ with the whole chain of length n at configuration $\{\varepsilon_i\}_1^n$. For the Kac model this is proportional to $\sum_{i=1}^n \varepsilon_i \lambda^i \cos \varphi i$. In the thermodynamical limit $n \to \infty$ the symbolic interaction function has a pronounced self-similar structure (see Fig. 2) whose fractal properties depend on the phase φ. If φ is a rational multiple of π, i.e. $\varphi = p\pi/q$ (p, q relative primes), the symbolic interaction function turns out to be a fractal which cannot be characterized by a single scaling factor. Its hierarchical organization consists of identical blocks of q different scalings. In other words, the same scaling factor appears on each qth level of the hierarchy.

Although the value of the energy constant E_0 is irrelevant for the spin chain, it turns out to play a crucial role in the associated map. Only at sufficiently large values of E_0 can a meaningful map be found, and there is only one value where the map exhibits permanent chaotic behaviour. Above this value the map possesses a gap and generates transient chaos [6]. Below it, the branches of the map start to overlap and the dynamics can be defined only in a random sense (see e.g. [7]).

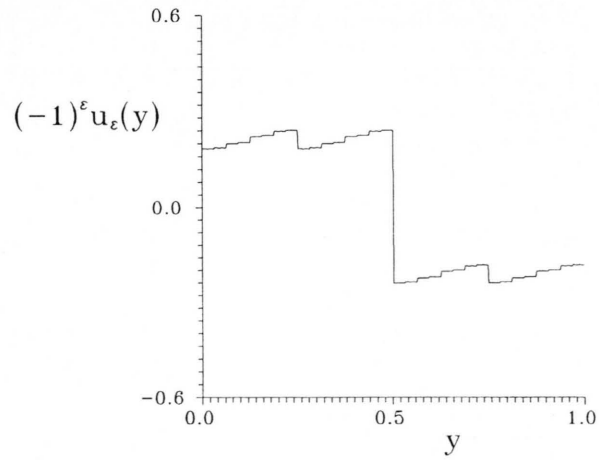

Fig. 2. Symbolic interaction function $u_\varepsilon(y) \sim \varepsilon \sum_i \varepsilon_i J(i)$ with J defined via (2) plotted at $|\lambda| = 0.3$ and $\varphi = \pi/4$. In the hierarchical construction of this graph the same scaling factor appears at each 4th level.

We have demonstrated that maps associated with the Kac model via the generalized FP operator (4) are *not* smoothly differentiable. Moreover, we believe that spin chains with smoothly varying two-body potentials always lead to such type of maps. In order to obtain a smooth map we have to include models with different types of multispin interactions. The smoothness of the associated map is a special property, and maps with fractal features will be found in most of the cases. This seems to be a relation not yet pointed out in the literature before.

As another new aspect, we stress that the above construction of associated maps can be applied to any arbitrary type of Ising chains with two-body interactions even if a simple functional representation of the transfer operator is not available. Models with interaction energy decaying with a polynomial expression of the distance (e.g. $J(x)=1/x^{\alpha}$, $\alpha>0$) lie in this class. Then a phase transition occurs for $\alpha<2$ [8]. In our construction the symbolic interaction function has then the form $u_{\varepsilon}(y)\sim\sum_{i=1}^{n}\varepsilon_i/i^{\alpha}$. For $\alpha>1$ and parallel configuration $\varepsilon_i\equiv 0$ or 1 (equilibrium configuration for $\beta\to\infty$) the length of the leftmost and rightmost cylinders scales as $\ln l(n)\sim\sum_{i=1}^{n}1/i^{\alpha-1}$ [2]. The above sum diverges logarithmically for $\alpha=2$ as $n\to\infty$. Then cylinder lengths near to the two fixed points show a power-law scaling in n. The associated map is thus nonhyperbolic. The reason for this *nonhyperbolicity* can be the marginal stability of one of the periodic orbits of the map that can be also manifested by an *intermittent* dynamics. This possible connection between Ising chains with long-range forces and intermittency might deserve further more detailed studies. For the complex extension of such models ($J(x)=x^{-\alpha}\cos\delta x$) the symbolic interaction function is proportional to $\sum_{i=1}^{n}\varepsilon_i i^{-\alpha}\cos\delta i$. Analizing the scaling of the cylinders we expect a sensitive δ-dependence for $\alpha\leq 2$, which would also be reflected by a nonanalyticity of the free energy.

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[1] M. Kac, Phys. Fluids **2,** 8 (1959); Brandeis University Summer Institute in Theoretical Physics, Gordon-Beach, New York 1966, Vol. 1.
[2] Á. Péntek, Z. Toroczkai, D. H. Mayer, and T. Tél, Phys. Rev. E **49,** 2026 (1994).
[3] C. Kittel, Solid State Phys. **22,** 1 (1968).
[4] D. H. Mayer, The Ruelle-Araki Transfer Operator in Classical Statistical Mechanics, Lecture Notes in Physics, Vol. **123,** Springer-Verlag, Berlin 1980.
[5] M. M. Feigenbaum, I. Procaccia, and T. Tél, Phys. Rev. A **39,** 5359 (1989).
[6] T. Tél, in: Directions in Chaos (Hao Bai-lin, ed.), World Scientific, Singapore 1990.
[7] G. Györgyi and P. Ruján, J. Phys. C **17,** 4207 (1984).
[8] C. J. Tompson, Mathematical Statistical Mechanics, Macmillan, New York 1971; in: Phase Transition and Critical Phenomena (C. Domb and M. Green, eds.), Academic, London 1972, Vol. 1.